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Elastic and sound orthonormal beams and localized fields in linear media: IV. Superpositions of sound plane waves in an ideal liquid

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Abstract

Superpositions of sound time-harmonic plane waves in an ideal liquid, defined by the spherical harmonics, are considered. By way of illustration we present unique families of orthonormal sound beams and localized fields. The obtained solutions describe fields having a very small (about several wavelengths) and clearly defined core region with maximum intensity of field oscillations. It is shown that, as in the case of localized fields in an elastic solid, there exist three families of localized sound fields in an ideal liquid (sound storms, whirls and tornadoes). The main types of the sound fields presented are illustrated by calculating fields, energy densities and energy fluxes.

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1. Introduction

Scalar, vector and tensor plane-wave superpositions, defined by a given set of orthonormal scalar functions on a two- or three-dimensional beam manifold $\mathcal{B}$ [1], are exact solutions of wave equations in linear media and/or free space. It was shown [1] that among such superpositions are included, in particular, orthonormal beams and other specific fields such as three-dimensional standing waves, moving and evolving whirls. The proposed formalism was initially developed for electromagnetic [1–3] and weak gravitational [3] fields and then extended [4] to elastic and sound fields. In the previous two papers [5, 6], we applied it to elastic fields in an isotropic medium, composed from longitudinal and transverse harmonic plane waves (eigenwaves), respectively. In the concluding paper of the current series, we consider sound fields in an ideal liquid.

As in the papers [5, 6], we confine here our illustrations to time-harmonic fields defined by the spherical harmonics Y_j^s as

$$\boldsymbol{W}_j^s(\boldsymbol{r}, t) = \exp(-\mathrm{i}\omega t) \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\theta_2} \exp[\mathrm{i}\boldsymbol{r} \cdot \boldsymbol{k}(\theta, \varphi)] Y_j^s(\theta, \varphi) \nu(\theta, \varphi) \boldsymbol{W}(\theta, \varphi) \sin\theta \, \mathrm{d}\theta. \quad (1)$$

We use the beam state function [4] $\nu = \nu(\theta, \varphi)$ to obtain a complete system of orthonormal sound beams formed from eigenwaves with unit wave normals given as

$$\hat{\boldsymbol{k}} = \boldsymbol{k}/k = \boldsymbol{e}_r = \sin\theta'(\boldsymbol{e}_1 \cos\varphi + \boldsymbol{e}_2 \sin\varphi) + \boldsymbol{e}_3 \cos\theta' \tag{2}$$

where $\boldsymbol{e}_r$ is the radial basis vector of the spherical coordinate system, $(\boldsymbol{e}_i)$ are the Cartesian basis vectors, $\theta' = \kappa_0\theta$ and parameter κ_0 satisfies the condition $0 < \kappa_0 \leqslant 1$. These eigenwaves propagate in the solid angle $\Omega = 2\pi(1 - \cos\kappa_0\theta_2)$. For other types of sound field, ν is a constant.

In the linear approximation, sound fields are described by the variations of pressure $p' = p - p_0$ and the density $\varrho' = \varrho - \varrho_0$, which are far less than the equilibrium values p_0 and ϱ_0. The velocity $\boldsymbol{v}$ of fluid particles is far less than the sound velocity c_0. Although a sound field in an ideal liquid is a typical example of scalar fields, when forming a set of orthonormal beams, we must use the two-component vector amplitude [4, 7]

$$\boldsymbol{W}(\theta, \varphi) \equiv \begin{pmatrix} p' \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \cos\theta'/(c_0\varrho_0) \end{pmatrix} \tag{3}$$

where $v_3 = \boldsymbol{e}_3 \cdot \boldsymbol{v}$ is the normal component of the velocity $\boldsymbol{v}$ of a fluid particle.

The outline of the paper is as follows. In section 2, we present two different types of orthonormal sound beam. Three families of localized sound fields are treated in section 3. Concluding remarks are made in section 4.

2. Orthonormal beams

2.1. Orthonormal beams with $\theta_2 = \pi/2$, $\kappa_0 = 1$, and $\Omega = 2\pi$

As with other physical fields (electromagnetic [1–3], weak gravitational [3] and elastic [5, 6]), the set of sound beams $\boldsymbol{W}_j^s$ (1) with $\theta_2 = \pi/2$ and $\kappa_0 = 1$ $(\theta' = \theta)$ consists of two separate orthonormalized subsets defined by the spherical harmonics Y_j^s with even and odd j, respectively. They are formed from eigenwaves propagating into a solid angle $\Omega = 2\pi$. As a consequence, the orthonormalizing function [4] $\nu = \nu(\theta, \varphi)$ reduces to a constant upon substitution of the amplitude function $\boldsymbol{W}$ (3), and we obtain

$$p' = \nu_3 \mathrm{e}^{\mathrm{i}(s\psi - \omega t)} I_j^{ss}[1] \qquad \nu_3 = \frac{2}{\lambda}\sqrt{\varrho_0 c_0 N_Q} \tag{4}$$

$$v_3 = \frac{\nu_3}{c_0\varrho_0} \mathrm{e}^{\mathrm{i}(s\psi - \omega t)} I_j^{ss}[\cos] \tag{5}$$

where $k = 2\pi/\lambda = \omega/c_0$, and N_Q is the normalizing constant [4]. Hereafter, we extensively use complex functions $I_j^{sm}[f] = I_j^{sm}[f](r, \gamma)$ and real functions $J_{jp}^{sm}[f] = J_{jp}^{sm}[f](r, \gamma)$, related as

$$I_j^{sm}[f] = \mathrm{i}^{|m|}(J_{j0}^{sm}[f] + \mathrm{i}J_{j1}^{sm}[f]). \tag{6}$$

These functions are defined by the spherical harmonic $Y_j^s = Y_j^s(\theta, \varphi)$, an integer m and a scalar function $f = f(\theta)$. For any given f, they are functions of r and γ, where r, γ and ψ are the spherical coordinates of the point with radius vector $\boldsymbol{r}$. However, they are functionals regarding f at fixed r and γ. The definitions and the properties of these functions are presented in [1, 4]. When it cannot cause a misunderstanding, we omit the arguments (r, γ).

Using the Euler equation [7], we find the velocity field

$$\boldsymbol{v} = \frac{\nu_3}{c_0\varrho_0} \mathrm{e}^{\mathrm{i}(s\psi - \omega t)} \left\{ \boldsymbol{e} I_j^{ss-1}[\sin] + \boldsymbol{e}^* I_j^{ss+1}[\sin] + \boldsymbol{e}_3 I_j^{ss}[\cos] \right\} \tag{7}$$

where

$$e = (e_R + \mathrm{i}e_A)/2 \tag{8}$$

$$e_R = e_1 \cos\psi + e_2 \sin\psi \qquad e_A = -e_1 \sin\psi + e_2 \cos\psi \tag{9}$$

$$r = Re_R + ze_3 \qquad R = r\sin\gamma \qquad z = r\cos\gamma. \tag{10}$$

Here, R, ψ and z are the cylindrical coordinates of the point with radius vector r and e_R and e_A are the radial and azimuthal basis vectors, respectively.

The time average sound energy density w_3 and energy flux vector S are given by

$$w_3 = w_1 + w_2 \tag{11}$$

$$w_1 = \frac{\varrho_0}{4}|v|^2 = w_0 \sum_{p=0}^{1} \left\{ \frac{1}{2}\big(J_{jp}^{ss-1}[\sin]\big)^2 + \frac{1}{2}\big(J_{jp}^{ss+1}[\sin]\big)^2 + \big(J_{jp}^{ss}[\cos]\big)^2 \right\} \tag{12}$$

$$w_2 = \frac{|p'|^2}{4\varrho_0 c_0^2} = w_0 \sum_{p=0}^{1} \big(J_{jp}^{ss}[1]\big)^2 \tag{13}$$

$$S = \tfrac{1}{2}\Re\left(p'v^*\right) = S_0\big(S'_R e_R + S'_A e_A + S'_N e_3\big) \tag{14}$$

$$S'_R = \sum_{p=0}^{1}(-1)^p J_{j1-p}^{ss}[1]\big\{\beta(-s)J_{jp}^{ss-1}[\sin] + \beta(s)J_{jp}^{ss+1}[\sin]\big\} \tag{15}$$

$$S'_A = \sum_{p=0}^{1} J_{jp}^{ss}[1]\big\{\beta(s)J_{jp}^{ss+1}[\sin] - \beta(-s)J_{jp}^{ss-1}[\sin]\big\} \tag{16}$$

$$S'_N = 2\sum_{p=0}^{1} J_{jp}^{ss}[\cos]J_{jp}^{ss}[1] \tag{17}$$

where

$$\beta(s) = \begin{cases} -1 & (s = -1, -2, \ldots) \\ 1 & (s = 0, 1, 2, \ldots) \end{cases} \tag{18}$$

$S_0 = N_Q/\lambda^2$ and $w_0 = S_0/c_0$. Both energy densities w_i ($i = 1, 2, 3$) and cylindrical components S'_R, S'_A and S'_N of the normalized energy flux vector $S' = S/S_0$ are independent of the azimuthal angle. All these parameters are symmetric functions of z. For the beams defined by the spherical harmonics Y_3^0 and Y_3^1, the energy density distribution in a meridional plane is depicted in figure 1. The beam defined by the spherical harmonic Y_3^0 is more localized in the radial directions than that defined by Y_3^1. For $s = 0$, the maximum values of w_3 are reached exactly at the z axis at the points $z' = \pm 0.8$. For $s = 1, 2$ and 3, peaks become lower and move away from this axis.

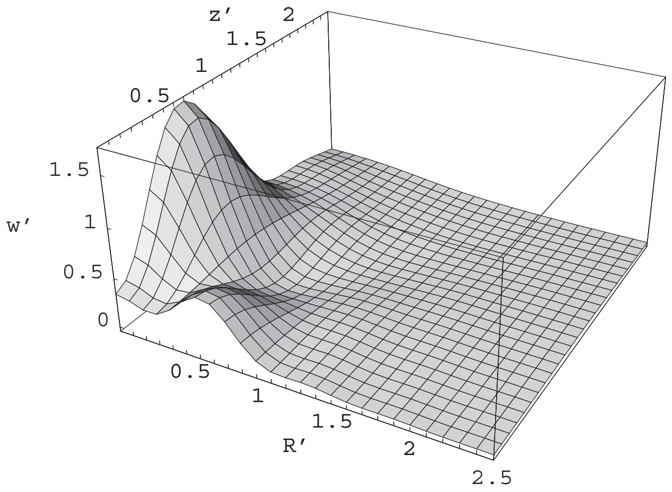
The zonal spherical harmonic Y_j^0 defines the sound beam with the velocity vectors lying in the meridional planes as

$$v = \frac{v_3}{c_0\varrho_0}\mathrm{e}^{-\mathrm{i}\omega t}\big\{e_R I_j^{01}[\sin] + e_3 I_j^{00}[\cos]\big\}. \tag{19}$$

2.2. *Orthonormal beams with $\theta_2 = \pi$, $\kappa_0 \leqslant 1/2$, and $\Omega \leqslant 2\pi$*

As in the case of the elastic beams [5, 6] with the same parameters $\theta_2 = \pi/2$ and $\kappa_0 = 1$, the sound fields presented above can be grouped into two separate sets of orthonormal beams, defined by the spherical harmonics Y_j^s with even and odd j, respectively.

To obtain a complete system of orthonormal beams [1, 4], defined by the whole set of spherical harmonics, it is necessary to set $\theta_2 = \pi$ and $\kappa_0 \leqslant 1/2$. This gives a unified



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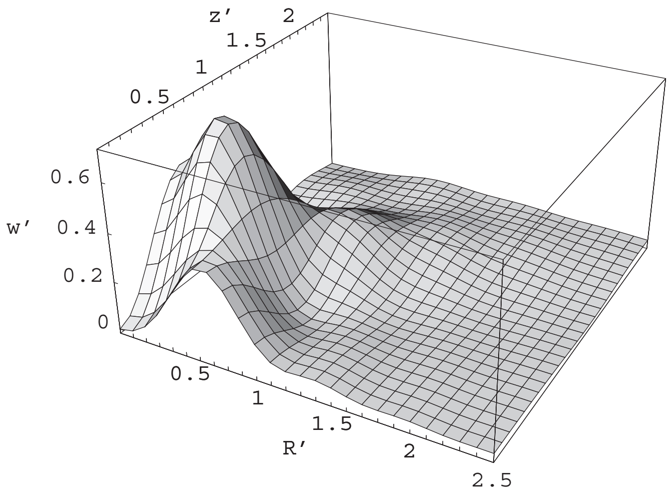

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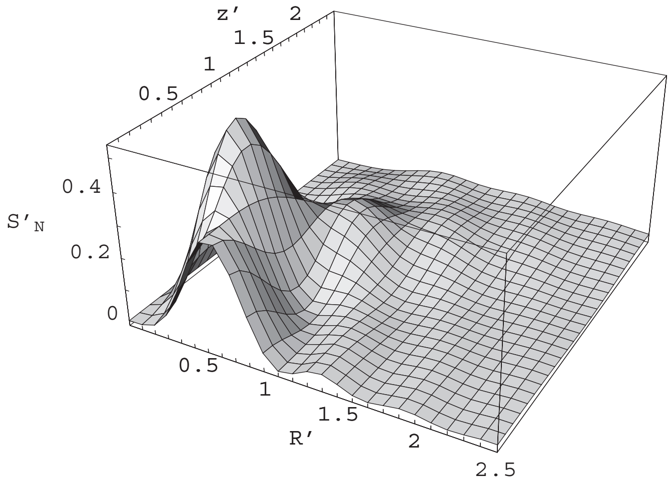
Figure 1. Normalized energy density $w' = w_3/w_0$ of sound beams as a function of cylindrical coordinates $R' = R/\lambda$ and $z' = z/\lambda$; $\theta_2 = \pi/2$; $\kappa_0 = 1$; $\Omega = 2\pi$; $j = 3$; (*a*) $s = 0$; (*b*) $s = 1$.

system of orthonormal sound beams formed from eigenwaves propagating in the solid angle $\Omega = 2\pi(1 - \cos\kappa_0\pi) \leqslant 2\pi$. In this case, the beam manifold is the unit sphere ($\mathcal{B} = S^2$); the orthonormalizing function [4] is of the form

$$\nu(\theta) = \frac{1}{\lambda}\sqrt{\frac{2\kappa_0\varrho_0 c_0 N_Q \sin\kappa_0\theta}{\sin\theta}}. \tag{20}$$

The divergence of beams depends on parameter κ_0. The smaller is κ_0, the smaller is the solid angle Ω in which propagate beam-forming eigenwaves. At $\kappa_0 = 1/2$ or $\kappa_0 \approx 1/2$, this family of orthonormal sound beams has a pronounced core region. Beams with $s \neq 0$ have spiral energy fluxes in the core and resemble sound tornadoes. Figure 2 illustrates S'_{N} for both



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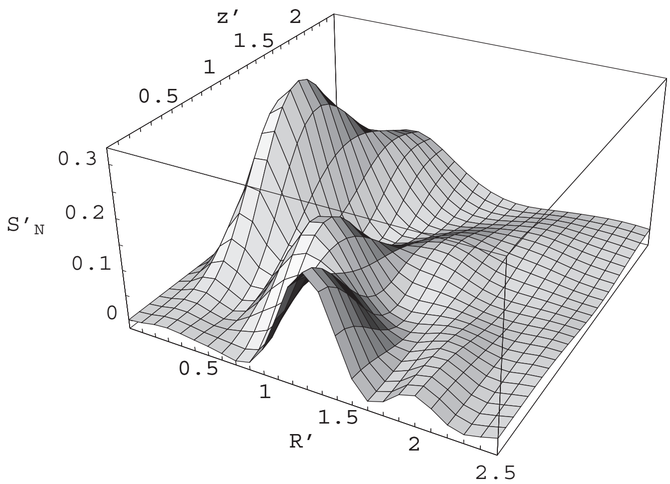

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Figure 2. Normal component S'_N of the normalized energy flux vector of sound beams as a function of $R' = R/\lambda$ and $z' = z/\lambda$; $\Omega = 2\pi$; $j = 3$; $s = 1$ (*a*) $\theta_2 = \pi/2$; $\kappa_0 = 1$; (*b*) $\theta_2 = \pi$; $\kappa_0 = 0.5$.

types of orthonormal beam defined by the same spherical harmonic Y_3^1. Although these beams are composed of plane waves propagating in the same solid angle $\Omega = 2\pi$, they have quite different spatial distributions of energy densities and energy fluxes. In particular, in the plane $z = 0$, as well as in any other cross-section, their main peaks of S'_N are situated in different domains.

3. Localized fields

As in the case of elastic fields [5, 6], localized time-harmonic sound fields (sound storms, whirls and tornadoes) are described by $\boldsymbol{W}_j^s$ (1) with $\pi/2 \leqslant \theta_2 \leqslant \pi$ and $\kappa_0 = 1$ ($\theta' = \theta$). The amplitude function $\boldsymbol{W}(\theta, \varphi)$ for these fields is determined by expression (3). As before, we assume that the beam state function $\nu = \nu(\theta, \varphi)$ reduces to a constant.

3.1. Sound storms and whirls

Substitution of $\boldsymbol{W}$ (3) in equation (1) with $\theta_2 = \pi$ results in the three-dimensional standing wave with the sound pressure field

$$p' = \mathrm{i}^{|s|+p}\sqrt{2}\nu_3 \mathrm{e}^{\mathrm{i}(s\psi-\omega t)} J_{jp}^{ss}[1] \tag{21}$$

and the velocity field

$$\boldsymbol{v} = \mathrm{i}^{|s|+q}\frac{\sqrt{2}\nu_3}{c_0\varrho_0}\mathrm{e}^{\mathrm{i}(s\psi-\omega t)}\left\{\boldsymbol{e}\,(-1)^p\beta(-s)J_{jp}^{ss-1}[\sin] + \boldsymbol{e}^*(-1)^p\beta(s)J_{jp}^{ss+1}[\sin] + \boldsymbol{e}_3 J_{jq}^{ss}[\cos]\right\}. \tag{22}$$

The time average energy densities w_1, w_2 and the sound energy flux vector field $\boldsymbol{S} = S_0 S'_{\mathrm{A}}\boldsymbol{e}_{\mathrm{A}}$ are given by

$$\frac{w_1}{w_0} = \left(J_{jp}^{ss-1}[\sin]\right)^2 + \left(J_{jp}^{ss+1}[\sin]\right)^2 + 2\left(J_{jq}^{ss}[\cos]\right)^2 \tag{23}$$

$$\frac{w_2}{w_0} = 2\left(J_{jp}^{ss}[1]\right)^2 \tag{24}$$

$$S'_{\mathrm{A}} = 2J_{jp}^{ss}[1]\left\{\beta(s)J_{jp}^{ss+1}[\sin] - \beta(-s)J_{jp}^{ss-1}[\sin]\right\} \tag{25}$$

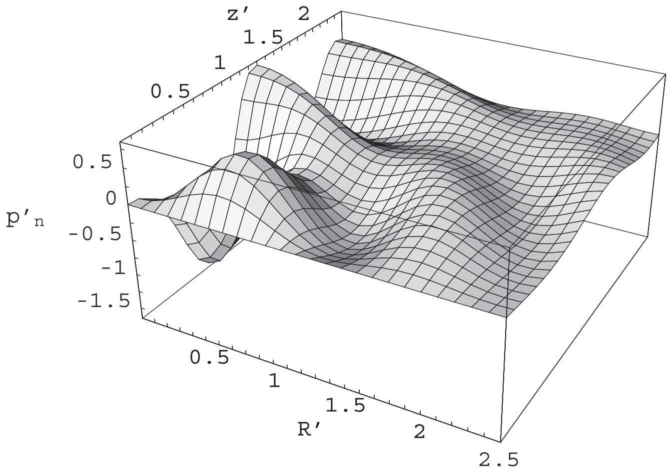
where $p = 1 - q = 0$ if $j + |s|$ is even, and $p = 1 - q = 1$ if $j + |s|$ is odd. The definitions of w_1, w_2, and $\boldsymbol{S}$ are given by equations (12)–(14).

Sound storms with identically vanishing energy flux vector $\boldsymbol{S}$ are defined by the zonal spherical harmonics ($s = 0$). Other spherical harmonics ($s \neq 0$) specify the sound whirls with azimuthal energy fluxes. The storm and the whirl, defined by the spherical harmonics Y_3^0 and Y_3^2, respectively, are illustrated in figure 3. The depicted parameters p'_n and S'_{A} are independent of ψ; i.e., the fields are axially symmetrical with respect to the z axis. For the sound storm under consideration, the intensity of pressure oscillations rapidly decreases in all directions with major oscillation peaks located along the z axis. In the storm centre as well as in the entire plane $z = 0$, this intensity is identically zero. The spatial distribution of the azimuthal energy fluxes for the sound whirls and the whirls composed of longitudinal elastic eigenwaves are similar in appearance (see figure 4 in part II [5] and figure 3 in this paper).

3.2. Tornadoes

To obtain a solution describing a localized sound field with spiral energy flux lines (sound tornado), it is sufficient to set $s \neq 0$ and $\pi/2 < \theta_2 < \pi$ in equation (1). The sound tornadoes have a pronounced core region and intensive azimuthal energy fluxes similar to those of the corresponding whirls (see figure 3). However, the radial and the normal components of the time average energy flux vector are not vanishing in this case (see figure 4) even though they are much smaller than the azimuthal one. All these cylindrical components are independent of the azimuthal angle ψ. As a result, the circular energy flux lines, typical for whirls, transform into spiral lines with the step specified by the ratio between S'_{N} and S'_{A} and the changing radius in domains with non-vanishing S'_{R} (see also [3, 5]). This is why we refer to these unique localized fields as sound tornadoes.

For the fields defined by the zonal spherical harmonics ($s = 0$), energy flux lines lie in meridional planes.



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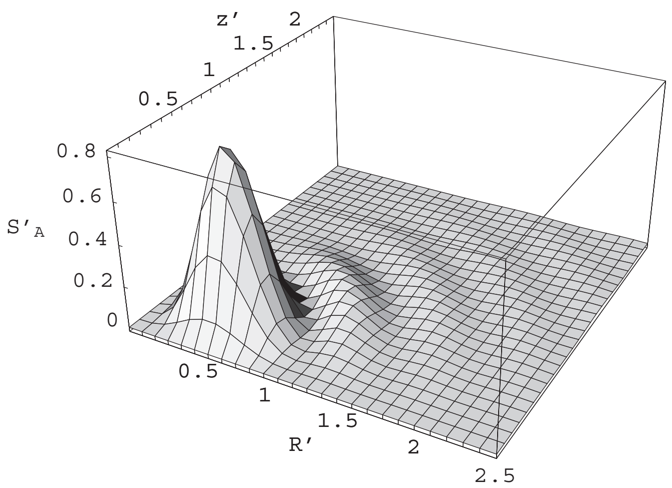

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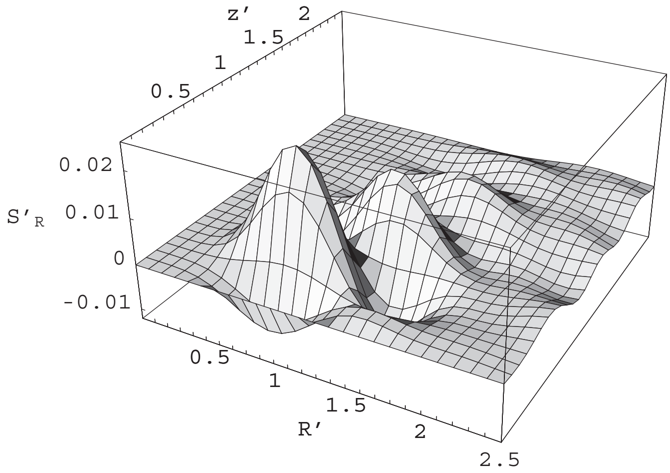
Figure 3. (*a*) Normalized instantaneous pressure field $p'_n = (\mathrm{Re}\, p')/p_n$ ($p_n = 2c_0\sqrt{\varrho_0 w_0}$) of a sound storm as a function of $R' = R/\lambda$ and $z' = z/\lambda$; $\theta_2 = \pi$; $\kappa_0 = 1$; $\Omega = 4\pi$; $j = 3$; $s = 0$; $\omega t = \pi/4$. (*b*) Azimuthal component S'_{A} of the normalized energy flux vector of a sound whirl as a function of R' and z'; $\theta_2 = \pi$; $\kappa_0 = 1$; $\Omega = 4\pi$; $j = 3$; $s = 2$.

4. Conclusion

In this paper, superpositions of sound time-harmonic plane waves in an ideal liquid, defined by the spherical harmonics, are treated. Unique solutions of the wave equation, describing families of orthonormal sound beams and specific localized fields are obtained. It is shown that, as in the case of localized fields in an elastic solid, there exist two different types of sound orthonormal beam, and three families of localized fields in an ideal liquid (sound storms, whirls and tornadoes).



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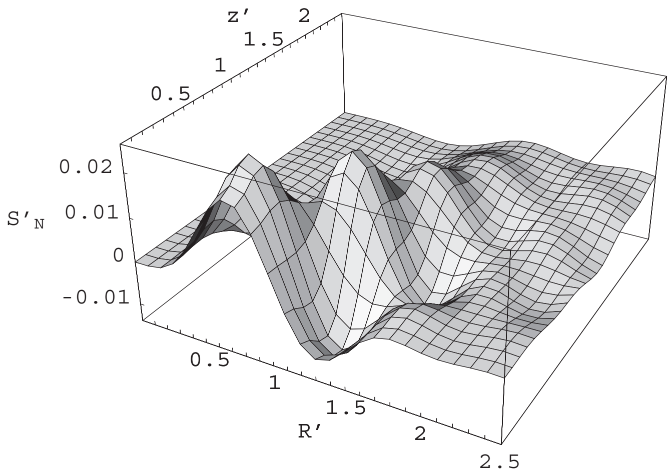

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Figure 4. (*a*) Radial $S'_{\rm R}$ and (*b*) normal $S'_{\rm N}$ components of the normalized energy flux vector of a sound tornado as a function of $R' = R/\lambda$ and $z' = z/\lambda$; $\theta_2 = 5\pi/6$; $\kappa_0 = 1$; $\Omega = 2\pi(1 + \sqrt{3}/2)$; $j = 3$; $s = 2$.

The solutions, obtained in the current series of papers, illustrate both similarities of and distinctions between scalar (sound waves), vector (radial displacement vector for longitudinal elastic waves, meridional and azimuthal displacement vectors for transverse elastic waves) and tensor (deformation and stress tensors) plane-wave superpositions defined by the same set of functions, such as the spherical harmonics. The elastic and sound fields presented in this series are defined by equations of the form (1). They can be classified as follows.

(1) Orthonormal beams with $\theta_1 = 0$, $\theta_2 = \pi/2$, $\kappa_0 = 1$ and $\Omega = 2\pi$. They are formed from plane waves propagating into a given half space. In this case, the beam manifold is the northern hemisphere $S^2_{\rm N}$ of the unit sphere S^2 ($\mathcal{B} = S^2_{\rm N}$).

(2) Orthonormal beams with $\theta_1 = 0$, $\theta_2 = \pi$ and $\kappa_0 \leqslant 1/2$. For these beams, the beam manifold is the unit sphere ($\mathcal{B} = S^2$), and $\Omega = 2\pi(1 - \cos\kappa_0\pi) \leqslant 2\pi$.
(3) Three-dimensional standing waves with $\theta_1 = 0$, $\theta_2 = \pi$, $\kappa_0 = 1$, $\mathcal{B} = S^2$ and $\Omega = 4\pi$. They are formed from plane waves of all possible propagation directions. The family of these waves consists of storms, defined by the zonal spherical harmonics Y_j^0, and whirls defined by the other Y_j^s ($s \neq 0$).
(4) Localized fields with $\theta_1 = 0$, $\pi/2 < \theta_2 < \pi$, $\kappa_0 = 1$ and $2\pi < \Omega < 4\pi$. These fields include tornadoes, which are defined by Y_j^s with $s \neq 0$.

As in the case of similar electromagnetic and weak gravitational fields [1–3], by integrating the presented time-harmonic elastic and sound localized fields with respect to the frequency as

$$\breve{\boldsymbol{W}}_j^s(\boldsymbol{r}, t) = \frac{1}{2\Delta\omega}\int_{\omega_-}^{\omega_+} \boldsymbol{W}_j^s(\boldsymbol{r}, t)\,\mathrm{d}\omega \tag{26}$$

where $\Delta\omega = (\omega_+ - \omega_-)/2 \ll (\omega_+ + \omega_-)/2$, one can also obtain finite-energy evolving sound storms, whirls and tornadoes.

In this series of papers, the elastic and sound fields are treated in the linear approximation. In a weakly nonlinear medium, the presented solutions can be used as initial approximations to solve the corresponding nonlinear wave equation by an iteration method. Alternatively, one can use these solutions to compose an evolving converging wave that is sufficiently weak at the initial stage to be treated in the linear approximation, i.e. at sufficiently small value of N_Q. This wave can be used to set initial conditions for a nonlinear equation describing the further evolution of this converging wave.

To realize localized-wave effects, an array that has independently addressable elements is required [8]. The effect of localized wave transmission has been verified with acoustic experiments using ultrasound in water and independently addressable, finite-sized arrays by Ziolkowski *et al* [9]. It seems plausible that similar arrays with sufficiently large number of elements can be used to radiate the beams presented in this series of papers.

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